

Symmetry in the Sieve: A Heuristic Link Between Twin Primes and Goldbach Partitions

Suqin Ao

Edmonton, AB Canada

ao.suqin@gmail.com

Lucy Lu

Edmonton, AB Canada

lucylu@ualberta.net

Abstract

The distribution of prime numbers exhibits a tension between randomness and rigid arithmetic structure. In this paper, we utilize a modified Sieve of Eratosthenes to explore the connection between two of the most famous problems in number theory: the Twin Prime Conjecture and the Goldbach Conjecture. By analyzing the sieving density, we demonstrate that the count of twin primes and the count of Goldbach partitions share a fundamental structural link. We derive a heuristic formula suggesting that the number of Goldbach partitions $\eta(2n)$ grows in proportion to the density of twin primes, denoted by $\eta(2n) \sim H\pi_2(n)$. This framework offers a fresh perspective on why these configurations of primes are not merely accidental, but statistically inevitable consequences of sieve theory.

Contents

1	Introduction	2
1.1	Approximating the Prime Density	2
2	The Twin Prime Sieve	2
3	The Goldbach Symmetry	3
3.1	Heuristic Derivation	4
4	Conclusion	5
A	Numerical Verification	6

1 Introduction

The prime numbers serve as the "atoms" of arithmetic. While their individual occurrence appears random, their asymptotic distribution follows the well-known Prime Number Theorem. However, questions regarding the distance between primes (Twin Primes) and the sum of primes (Goldbach Conjecture) remain elusive.

A *prime number* is a natural number greater than 1 having no positive divisors other than 1 and itself. We employ standard notation: $\mathbb{N}$ for natural numbers, $\ln x$ for the natural logarithm, and $\pi(x)$ for the prime-counting function.

The classical tool for identifying primes is the *Sieve of Eratosthenes* [2]. For a large positive number n , let $S = \{m \in \mathbb{N} \mid 1 < m \leq n\}$. The set of primes in S , denoted $P[S]$, is obtained by removing multiples of smaller primes:

$$P[S] = S - \bigcup_{p \leq \sqrt{n}} S_p, \quad (1)$$

where $S_p = \{mp \mid m \in \mathbb{N}, p \leq m \leq n/p\}$.

1.1 Approximating the Prime Density

To understand the density of primes, we utilize Euler's totient function, $\phi(n)$. For a large x , let N be the primorial, defined as the product of all primes $p \leq \sqrt{x}$. The number of integers coprime to N describes the "potential" primes remaining after sieving. The count of such numbers is given by:

$$\phi(N) = N \prod_{p \leq \sqrt{x}} \left(1 - \frac{1}{p}\right). \quad (2)$$

From [2], utilizing Mertens' theorems and the Euler constant γ , we approximate the prime counting function as:

$$\pi(x) \sim \frac{e^\gamma}{2} x \prod_{p \leq \sqrt{x}} \left(1 - \frac{1}{p}\right) \sim \frac{x}{\ln x} \sim Li(x). \quad (3)$$

This establishes the baseline density of primes. Our objective is to extend this sieving logic to *pairs* of primes.

2 The Twin Prime Sieve

Twin primes are pairs $(p, p+2)$ where both are prime. We can view this as a stricter sieve: a number p survives if it is not divisible by a prime q , and it *also* survives if $p+2$ is not divisible by q .

To find twin primes up to n , we must sieve the set S twice. First, we remove multiples of primes (standard Sieve). Second, we remove numbers m where $m+2$ is a multiple of a prime.

Mathematically, the remaining set $P_1 P[S]$ is:

$$P_1 P[S] = P[S] - \bigcup_{p \leq \sqrt{n+2}} S_p(2), \quad (4)$$

where $S_p(2)$ represents the set of numbers congruent to $-2 \pmod{p}$.

For $n = 100$, this double-sieve process leaves us with the set:

$$\{3, 5, 11, 17, 29, 41, 59, 71\}.$$

These correspond to the pairs $(3, 5), (5, 7), \dots, (71, 73)$.

Table 1 visually demonstrates this process. Cells marked in **Grey** are eliminated by the first sieve (standard composites). Cells in **Red** or **Green** represent numbers where the candidate p is prime, but $p+2$ is composite, eliminating the pair. The white cells represent the surviving Twin Primes.

s	n	n+2	s	n	n+2	s	n	n+2	s	n	n+2	s	n	n+2
		3	21	21	23	41	41	43	61	61	63	81	81	83
2	2	4	22	22	24	42	42	44	62	62	64	82	82	84
3	3	5	23	23	25	43	43	45	63	63	65	83	83	85
4	4	6	24	24	26	44	44	46	64	64	66	84	84	86
5	5	7	25	25	27	45	45	47	65	65	67	85	85	87
6	6	8	26	26	28	46	46	48	66	66	68	86	86	88
7	7	9	27	27	29	47	47	49	67	67	69	87	87	89
8	8	10	28	28	30	48	48	50	68	68	70	88	88	90
9	9	11	29	29	31	49	49	51	69	69	71	89	89	91
10	10	12	30	30	32	50	50	52	70	70	72	90	90	92
11	11	13	31	31	33	51	51	53	71	71	73	91	91	93
12	12	14	32	32	34	52	52	54	72	72	74	92	92	94
13	13	15	33	33	35	53	53	55	73	73	75	93	93	95
14	14	16	34	34	36	54	54	56	74	74	76	94	94	96
15	15	17	35	35	37	55	55	57	75	75	77	95	95	97
16	16	18	36	36	38	56	56	58	76	76	78	96	96	98
17	17	19	37	37	39	57	57	59	77	77	79	97	97	99
18	18	20	38	38	40	58	58	60	78	78	80	98	98	100
19	19	21	39	39	41	59	59	61	79	79	81	99	99	
20	20	22	40	40	42	60	60	62	80	80	82	100	100	

Table 1: Visualizing the Twin Prime Sieve (Composite filtering in Grey/Red/Green)

Alternatively, applying the sieve logic to the set of existing primes $P(100)$ yields Table 2, highlighting the surviving pairs more compactly.

Table 2: Sieve of Eratosthenes applied to $P(100)$

p	p+2		p	p+2		p	p+2		p	p+2		p	p+2
2	4		13	15		31	33		53	55		73	75
3	5		17	19		37	39		59	61		79	81
5	7		19	21		41	43		61	63		83	85
7	9		23	25		43	45		67	69		89	91
11	13		29	31		47	49		71	73		97	99

3 The Goldbach Symmetry

We now turn to the Goldbach Conjecture, which posits that every even number $2n > 2$ is the sum of two primes, $a + b = 2n$.

This problem is structurally the "mirror" of the twin prime problem. In the twin prime case, we look for a difference of 2. In the Goldbach case, we look for a sum of $2n$. The sieving process is remarkably similar. For a fixed n , we consider pairs (a, b) such that $a + b = 2n$. A pair survives if:

- a is not divisible by a prime p (standard sieve).
- b (which is $2n - a$) is not divisible by a prime p .

Tables 3 and 4 (labeled 5 and 6 in original datasets) illustrate this mechanism. We filter out pairs where either component is composite.

60=a+b			62=a+b			64=a+b			66=a+b			68=a+b		
k	a	b	k	a	b	k	a	b	k	a	b	k	a	b
1	1	59	1	1	61	1	1	63	1	1	65	1	1	67
3	3	57	3	3	59	3	3	61	3	3	63	3	3	65
5	5	55	5	5	57	5	5	59	5	5	61	5	5	63
7	7	53	7	7	55	7	7	57	7	7	59	7	7	61
9	9	51	9	9	53	9	9	55	9	9	57	9	9	59
11	11	49	11	11	51	11	11	53	11	11	55	11	11	57
13	13	47	13	13	49	13	13	51	13	13	53	13	13	55
15	15	45	15	15	47	15	15	49	15	15	51	15	15	53
17	17	43	17	17	45	17	17	47	17	17	49	17	17	51
19	19	41	19	19	43	19	19	45	19	19	47	19	19	49
21	21	39	21	21	41	21	21	43	21	21	45	21	21	47
23	23	37	23	23	39	23	23	41	23	23	43	23	23	45
25	25	35	25	25	37	25	25	39	25	25	41	25	25	43
27	27	33	27	27	35	27	27	37	27	27	39	27	27	41
29	29	31	29	29	33	29	29	35	29	29	37	29	29	39
			31	31	31	31	31	33	31	31	35	31	31	37
									33	33	33	33	33	35

Table 3: Goldbach Partitions Sieve

Surviving Pairs after Sieve (Goldbach Solutions)														
60 = p + q			62 = p + q			64 = p + q			66 = p + q			68 = p + q		
s	p	q	s	p	q	s	p	q	s	p	q	s	p	q
7	7	53	3	3	59	3	3	61	5	5	61	7	7	61
13	13	47	19	19	43	5	5	59	7	7	59	31	31	37
17	17	43	31	31	31	11	11	53	13	13	53			
19	19	41				17	17	47	19	19	47			
23	23	37				23	23	41	23	23	43			
29	29	31							29	29	37			

Table 4: Result of the Goldbach Sieve

3.1 Heuristic Derivation

The similarity in sieving leads to a striking probabilistic prediction. If we model the primes as distributed randomly with density $1/\ln n$, then the expected number of Goldbach pairs, $\eta(2n)$, should satisfy:

$$\eta(2n) \geq G \frac{n}{(\ln n)^2} > 1. \quad (5)$$

Furthermore, since the density of twin primes $\pi_2(n)$ is governed by the same "double sieve" logic, we propose the following asymptotic relation:

$$\eta(2n) \sim H \pi_2(n), \quad (6)$$

where H is a constant related to the specific divisors of n . This suggests that if the Twin Prime Conjecture is true (infinite pairs), the Goldbach Conjecture (infinite partition possibilities as n grows) follows from the same structural necessity.

4 Conclusion

In this paper, we have presented a sieve-theoretic framework connecting the distribution of twin primes to the Goldbach partitions. While the problem of rigorously bounding the error terms in sieve methods remains the central challenge of analytic number theory, our derivations show that the "main terms" of these distributions are structurally linked.

The numerical data (see Appendix A) consistently aligns with the theoretical expectation that $\eta(2n)$ grows in proportion to $n/(\ln n)^2$. This reinforces the likelihood that the Goldbach Conjecture is not just true, but is a necessary consequence of the random-yet-ordered nature of prime numbers.

Acknowledgments

We thank the referees for their careful reading and suggestions.

A Numerical Verification

The following table compares the actual Goldbach number $\eta(N)$ with our heuristic approximation g :

$$g = \frac{N/2}{(\ln(N/2))^2}$$

n	N=12n	$\eta(N)$	g	$\eta(N)-g$	N=12n+2	$\eta(N)$	g	$\eta(N)-g$	N=12n+4	$\eta(N)$	g	$\eta(N)-g$
1	12	1	1.87	-0.9	14	2	1.85	0.2	16	2	1.85	0.1
2	24	3	1.94	1.1	26	3	1.98	1.0	28	2	2.01	0.0

Table 5: Comparison of Goldbach number and approximation

References

- [1] George E. Andrews. *Number Theory*. Dover Publications, 1994.
- [2] G. H. Hardy and E. M. Wright. *An Introduction to the Theory of Numbers*. Oxford University Press, 2008.
- [3] Alphonse de Polignac. Recherches nouvelles sur les nombres premiers. *Comptes Rendus*, 29:397–401, 1849.
- [4] G. H. Hardy and J. E. Littlewood. Some problems of 'Partitio numerorum'; III: On the expression of a number as a sum of primes. *Acta Mathematica*, 44:1–70, 1923.
- [5] Christian Goldbach. Letter to L. Euler, June 7, 1742.